Fefferman's Inequality and Unique Continuation Property of Elliptic Partial Differential Equations

Preprint · May 2019

3 authors:

Nicky Kurnia Tumalun
Universitas Negeri Manado

Denny Ivanal Hakim
Bandung Institute of Technology

Hendra Gunawan
Bandung Institute of Technology

FEFFERMAN'S INEQUALITY AND APPLICATIONS IN ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

NICKY K. TUMALUN[1,*], DENNY I. HAKIM[2], and HENDRA GUNAWAN[3]

ABSTRACT. In this paper we prove Fefferman's inequalities associated to potentials belonging to a generalized Morrey space $L^{p,\varphi}$ or a Stummel class $\tilde{S}_{\alpha,p}$. Our results generalize and extend Fefferman's inequalities obtained in [2, 4, 10, 30]. We also show that the logarithmic of non-negative weak solution of second order elliptic partial differential equation, where its potentials are assumed in generalized Morrey spaces and Stummel classes, belongs to the bounded mean oscillation class. As a consequence, this elliptic partial differential equation has the strong unique continuation property. An example of an elliptic partial differential equation where its potential belongs to certain Morrey spaces or Stummel classes which does not satisfy the strong unique continuation is presented.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let $1 \leq p < \infty$ and $\varphi : (0,\infty) \to (0,\infty)$. The **generalized Morrey space** $L^{p,\varphi} := L^{p,\varphi}(\mathbb{R}^n)$, which was introduced by Nakai in [20], is the collection of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{L^{p,\varphi}} := \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{\varphi(r)} \int_{|x-y|<r} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

Note that $L^{p,\varphi}$ is a Banach space with norm $\|\cdot\|_{L^{p,\varphi}}$. If $\varphi(r) = 1$, then $L^{p,\varphi} = L^p$. If $\varphi(r) = r^n$, then $L^{p,\varphi} = L^\infty$. If $\varphi(r) = r^\lambda$ where $0 < \lambda < n$, then $L^{p,\varphi} = L^{p,\lambda}$ is the classical Morrey space introduced in [18].

We will assume the following conditions for φ which will be stated whenever necessary.

(1) There exists $C > 0$ such that

$$s \leq t \Rightarrow \varphi(s) \leq C\varphi(t). \tag{1.1}$$

We say φ **almost increasing** if φ satisfies this condition.

(2) There exists $C > 0$ such that

$$s \leq t \Rightarrow \frac{\varphi(s)}{s^n} \geq C\frac{\varphi(t)}{t^n}. \tag{1.2}$$

We say $\varphi(t)t^{-n}$ **almost decreasing** if $\varphi(t)t^{-n}$ satisfies this condition.

2010 *Mathematics Subject Classification.* 26D10 46E30 35J15.

Key words and phrases. Morrey spaces; Stummel classes; Fefferman's inequality; Strong unique continuation property.

*Corresponding author.

(3) For $1 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, there exists a constant $C > 0$ such that for every $\delta > 0$,
$$\int_{\delta}^{\infty} \frac{\varphi(t)}{t^{(n+1)-\frac{p}{2}(\alpha+1)}}\, dt \le C\delta^{\frac{p}{2}(1-\alpha)}. \tag{1.3}$$
This last condition for φ will be called **Nakai's condition**.

One can check that the function $\varphi(t) = t^{n-\alpha p}$, $t > 0$, satisfies all conditions (1.1), (1.2), and (1.3). Moreover, for a non-trivial example, we have the function $\varphi_0(t) = \log(\varphi(t)+1) = \log(t^{n-\alpha p}+1)$, $t > 0$, which satisfies all conditions above.

Let M be the **Hardy-Littlewood maximal operator**, defined by
$$M(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$
for every $f \in L^1_{\rm loc}(\mathbb{R}^n)$. The function $M(f)$ is called the **Hardy-Littlewood maximal function**. Notice that, for every $f \in L^p_{\rm loc}(\mathbb{R}^n)$ where $1 \le p \le \infty$, $M(f)(x)$ is finite for almost all $x \in \mathbb{R}^n$. Using Lebesgue Differentiation Theorem, we have
$$|f(x)| = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \le \lim_{r\to 0} M(f)(x) = M(f)(x), \tag{1.4}$$
for every $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and for almost all $x \in \mathbb{R}^n$. Furthermore, for every $f \in L^p_{\rm loc}(\mathbb{R}^n)$ where $1 \le p \le \infty$ and $0 < \gamma < 1$, the nonnegative function $w(x) := [M(f)(x)]^\gamma$ is an A_1 weight, that is,
$$M(w)(x) \le C(n,\gamma) w(x).$$
These maximal operator properties can be found in [11, 26].

We will need the following property about the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces $L^{p,\varphi}$ which is stated in [20, 21, 24], that is,
$$\|M(f)\|_{L^{p,\varphi}} \le C(n,p)\|f\|_{L^{p,\varphi}},$$
for every $f \in L^{p,\varphi}$, where $1 \le p < \infty$ and φ satisfies conditions (1.1) and (1.2). Note that in [20], the proof of this boundedness property relies on a condition about the integrability of $\varphi(t)t^{-(n+1)}$ over the interval (δ,∞) for every positive number δ.

Let $1 \le p < \infty$ and $0 < \alpha < n$. For $V \in L^p_{\rm loc}(\mathbb{R}^n)$, we write
$$\eta_{\alpha,p}V(r) := \sup_{x\in\mathbb{R}^n} \left(\int_{|x-y|<r} \frac{|V(y)|^p}{|x-y|^{n-\alpha}}\, dy \right)^{\frac{1}{p}}, \quad r > 0.$$
We call $\eta_{\alpha,p}V$ the **Stummel p-modulus** of V. If $\eta_{\alpha,p}V(r)$ is finite for every $r > 0$, then $\eta_{\alpha,p}V(r)$ is nondecreasing on the set of positive real numbers and satisfies
$$\eta_{\alpha,p}V(2r) \le C(n,\alpha)\, \eta_{\alpha,p}V(r), \quad r > 0.$$

The last inequality is known as the **doubling condition** for the Stummel $p-$modulus of V [28, p.550].

For each $0 < \alpha < n$ and $1 \leq p < \infty$, let

$$\tilde{S}_{\alpha,p} := \{V \in L^p_{\text{loc}}(\mathbb{R}^n) : \eta_{\alpha,p}V(r) < \infty \,\text{for all}\, r > 0\}$$

and

$$S_{\alpha,p} := \left\{V \in L^p_{\text{loc}}(\mathbb{R}^n) : \eta_{\alpha,p}V(r) < \infty \text{ for all}\, r > 0 \,\text{and}\, \lim_{r\to 0} \eta_{\alpha,p}V(r) = 0\right\}.$$

The set $S_{\alpha,p}$ is called a **Stummel class**, while $\tilde{S}_{\alpha,p}$ is called a **bounded Stummel modulus class**. For $p = 1$, $S_{\alpha,1} := S_\alpha$ are the Stummel classes which were introduced in [6, 23]. We also write $\tilde{S}_{\alpha,1} := \tilde{S}_\alpha$ and $\eta_{\alpha,1} := \eta_\alpha$. It was shown in [28] that $\tilde{S}_{\alpha,p}$ contains $S_{\alpha,p}$ properly. These classes play an important role in studying the regularity theory of partial differential equations (see [2, 3, 6, 26, 30] for example), and have an inclusion relation with Morrey spaces under some certain conditions [2, 23, 27, 28].

Now we state our results for Fefferman's inequalities:

Theorem 1.1. *Let $1 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, and φ satisfy conditions (1.1), (1.2), (1.3). If $V \in L^{p,\varphi}$, then*

$$\int_{\mathbb{R}^n} |u(x)|^\alpha |V(x)| dx \leq C\|V\|_{L^{p,\varphi}} \int_{\mathbb{R}^n} |\nabla u(x)|^\alpha dx \tag{1.5}$$

for every $u \in C_0^\infty(\mathbb{R}^n)$.

Theorem 1.2. *Let $1 \leq p < \infty$, $1 \leq \alpha \leq 2$, and $\alpha < n$. If $V \in \tilde{S}_{\alpha,p}(\mathbb{R}^n)$, then there exists a constant $C := C(n, \alpha) > 0$ such that*

$$\int_{B(x_0,r_0)} |V(x)|^p |u(x)|^\alpha \, dx \leq C[\eta_{\alpha,p}V(r_0)]^p \int_{B(x_0,r_0)} |\nabla u(x)|^\alpha \, dx,$$

for every ball $B_0 := B(x_0, r_0) \subseteq \mathbb{R}^n$ and $u \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp}(u) \subseteq B_0$.

Remark 1.3. The assumption that the function u belongs to $C_0^\infty(\mathbb{R}^n)$ in Theorem 1.1 and Theorem 1.2 can be weakened by the assumption that u has a weak gradient in a ball $B \subset \mathbb{R}^n$ and a compact support in B (see [29, p.480]).

In 1983, C. Fefferman [10] proved Theorem 1.1 for the case $V \in L^{p,n-2p}$, where $1 < p \leq \frac{n}{2}$. The inequality (1.5) is now known as **Fefferman's inequality**. Chiarenza and Frasca [4] extended the result [10] by proving Theorem 1.1 under the assumption that $V \in L^{p,n-\alpha p}$, where $1 < \alpha < n$ and $1 < p \leq \frac{n}{\alpha}$. By setting $\varphi(t) = t^{n-\alpha p}$ in Theorem 1.1, we can recover the result in [4] and [10].

For the particular case where $V \in \tilde{S}_2$, Theorem 1.2 is proved by Zamboni [30], and can be also concluded by applying the result Fabes *et al.* in [9, p.197] with an additional assumption that V is a radial function. Recently, this result is reproved in [2]. Although $\tilde{S}_\alpha \subset \tilde{S}_2$ whenever $1 \leq \alpha \leq 2$ [28, p.553], the authors still do not know how to deduce Theorem 1.2 from this result.

It must be noted that Theorems 1.1 and 1.2 are independent each other, which means that $L^{p,n-\alpha p}$, where $1 < \alpha < n$ and $1 < p \leq \frac{n}{\alpha}$, is not contained in $S_{\alpha,p}$.

Conversely, $S_{\alpha,p}$ is not contained in $L^{p,n-\alpha p}$. Indeed, if we define $V_1 : \mathbb{R}^n \to \mathbb{R}$ by the formula $V_1(y) := |y|^{-\alpha}$, then $V_1 \in L^{p,n-\alpha p}$, but $V_1 \notin \tilde{S}_{\alpha,p}$. For the function $V_2 : \mathbb{R}^n \to \mathbb{R}$ which is defined by the formula $V_2(y) := |y|^{-\frac{1}{p}}$, we have $V_2 \in \tilde{S}_{\alpha,p}$, but $V_2 \notin L^{p,n-\alpha p}$.

In order to apply Theorems 1.1 and 1.2, let us recall the following definitions. Let Ω be an open and bounded subset of $\mathbb{R}^n$. Recall that the **Sobolev space** $H^1(\Omega)$ is the set of all functions $u \in L^2(\Omega)$ for which $\dfrac{\partial u}{\partial x_i} \in L^2(\Omega)$ for all $i = 1, \ldots, n$, and is equipped by the **Sobolev norm**

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(\Omega)}.$$

The closure of $u \in C_0^\infty(\Omega)$ in $H^1(\Omega)$ under the Sobolev norm is denoted by $H_0^1(\Omega)$.

Define the operator L on $H_0^1(\Omega)$ by

$$Lu := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + Vu \tag{1.6}$$

where $a_{ij} \in L^\infty(\Omega)$, b_i $(i, j = 1, \ldots, n)$ and V are real valued measurable functions on $\mathbb{R}^n$. Throughout this paper, we assume that the matrix $a(x) := (a_{ij}(x))$ is symmetric on Ω and that the ellipticity and boundedness conditions

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \tag{1.7}$$

hold for some $\lambda > 0$, for all $\xi \in \mathbb{R}^n$, and for almost all $x \in \Omega$.

There are two assumptions on the potentials of the operator L (1.6) in this paper. We assume either:

$$\begin{cases} \varphi \text{ satisfies (1.1), (1.2), (1.3) } (1 < \alpha \le 2), \\ b_i^2 \in L^{p,\varphi},\ i = 1, \ldots, n, \\ V \in L^{p,\varphi} \cap L^2_{\text{loc}}(\mathbb{R}^n), \end{cases} \tag{1.8}$$

or,

$$\begin{cases} 1 \le \alpha \le 2, \\ b_i^2 \in \tilde{S}_\alpha,\ i = 1, \ldots, n, \\ V \in \tilde{S}_\alpha. \end{cases} \tag{1.9}$$

We say that $u \in H_0^1(\Omega)$ is a **weak solution** of the equation

$$Lu = 0 \tag{1.10}$$

if

$$\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}\psi + Vu\psi \right) dx = 0. \tag{1.11}$$

for all $\psi \in H_0^1(\Omega)$ (see the definition in [2, 7, 30]). Note that, for the case $\alpha = 2$, the equation (1.10) was considered in [2, 30]. If we choose $b_i = 0$ for all $i = 1, \ldots, n$, then (1.10) becomes the Schrödinger equation [3].

A locally integrable function f on $\mathbb{R}^n$ is said to be of **bounded mean oscillation** on a ball $B \subseteq \mathbb{R}^n$ if there is a constant $C > 0$ such that for every ball $B' \subseteq B$,

$$\frac{1}{|B'|}\int\limits_{B'} |f(y) - f_{B'}|dy \leq C.$$

We write $f \in BMO(B)$ if f is of bounded mean oscillation on B. Moreover, if $1 \leq \alpha < \infty$ and there is a constant $C > 0$ such that for every ball $B' \subseteq B$,

$$\left(\frac{1}{|B'|}\int\limits_{B'} |f(y) - f_{B'}|^{\alpha}dy\right)^{\frac{1}{\alpha}} \leq C,$$

we write $f \in BMO_{\alpha}(B)$. By using Hölder's inequality and the John-Nirenberg theorem (we refer to [22] for more detail about this John-Nirenberg Theorem), we can prove that $BMO_{\alpha}(B) = BMO(B)$.

As an application Theorems 1.1 and 1.2 to equation $Lu = 0$ (1.10), we have the following result.

Theorem 1.4. *Let $u \geq 0$ be a weak solution of $Lu = 0$ and $B(x, 2r) \subseteq \Omega$ where $r \leq 1$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} \left|\log(u+\delta) - \log(u+\delta)_{B(x,r)}\right|^{\alpha} dy \leq C,$$

for every $\delta > 0$.

Theorem 1.4 tells us that $\log(u + \delta) \in BMO_{\alpha}(B)$, where B is an open ball which is contained in Ω, and u is the non-negative weak solution of equation $Lu = 0$, given by (1.10). Under the assumptions that b_i^2 and V belong to $\tilde{S}_2$, Theorem 1.4 is obtained in [3] for the case Schrödinger equation ($b_i = 0$) and in [30, 2] for the case $Lu = 0$ ($b_i \neq 0$). To the best of our knowledge, the assumptions in (1.8) have never been used for proving Theorem 1.4 as well as the assumption $\alpha \in [1, 2)$ in (1.9).

Let $w \in L^1_{\text{loc}}(\Omega)$ and $w \geq 0$ in Ω. The function w is said to **vanish of infinite order** at $x_0 \in \Omega$ if

$$\lim_{r\to 0}\frac{1}{|B(x_0,r)|^k}\int\limits_{B(x_0,r)} w(x)dx = 0, \qquad \forall k > 0.$$

The equation $Lu = 0$, which is given in (1.10), is said to have the **strong unique continuation property** in Ω if for every nonnegative solution u which vanishes of infinite order at some $x_0 \in \Omega$, then $u \equiv 0$ in $B(x_0, r)$ for some $r > 0$. See this definition, for example in [12, 15, 16].

Theorem 1.4 gives the following result.

Corollary 1.5. *The equation $Lu = 0$ has the strong unique continuation property in Ω.*

This strong unique continuation property was studied by several authors. For example, Chiarenza and Garofalo in [4] discussed the Schrödinqer inequality of the form $Lu + Vu \geq 0$, where the potential V belongs to Lorentz spaces $L^{\frac{n}{2},\infty}(\Omega)$. For the differential inequality of the form $|\Delta u| \leq |V||u|$ where its potential also belong to $L^{\frac{n}{2}}(\Omega)$, see Jerison and Kenig [15]. Garofalo and Lin [12] studied the equation (1.10) where the potentials are bounded by certain functions.

Fabes *et al.* studied the strong unique continuation property for Schrödinqer equation $-\Delta u + Vu = 0$, where the assumption for V is radial function in S_2 [9]. Meanwhile, Zamboni [30] and Castillo *et al.* [2] also studied the equation (1.10) under the assumption that the potentials belong to S_2. At the end of this paper, we will give an example of Schrödinqer equation $-\Delta u + Vu = 0$ that does not satisfy the strong unique continuation property, where $V \in L^{p,n-4p}$ or $V \in \tilde{S}_\beta$ for all $\beta \geq 4$.

2. Proofs

In this section, we prove Fefferman's inequalities, which we state as Theorems 1.1 and 1.2 above. First, we start with the case where the potential belongs to a generalized Morrey spaces. Second, we consider the potential from a Stummel class. Furthermore, we present an inequality which is deduced from this inequality.

2.1. Fefferman's Inequality in Generalized Morrey Spaces.

We start with the following lemma for potentials in generalized Morrey spaces.

Lemma 2.1. *Let $1 < p < \infty$ and φ satisfy the conditions (1.1) and (1.2). If $1 < \gamma < p$ and $V \in L^{p,\varphi}$, then $[M(|V|^\gamma)]^{\frac{1}{\gamma}} \in A_1 \cap L^{p,\varphi}$.*

Proof. According to our discussion above, $[M(|V|^\gamma)]^{\frac{1}{\gamma}} \in A_1$. Using the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces, we have

$$\|[M(|V|^\gamma)]^{\frac{1}{\gamma}}\|_{L^{p,\varphi}} = \|M(|V|^\gamma)\|^{\frac{1}{\gamma}}_{L^{\frac{p}{\gamma},\varphi}} \leq C\|V\|_{L^{p,\varphi}} < \infty.$$

Therefore $[M(|V|^\gamma)]^{\frac{1}{\gamma}} \in L^{p,\varphi}$. □

Using Lemma 2.1, we obtain the following property.

Lemma 2.2. *Let φ satisfy the conditions (1.1), (1.2), and (1.3). If $V \in L^{p,\varphi}$, then*

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\, dy \leq C(n,\alpha,p)\|V\|^{\frac{1}{\alpha}}_{L^{p,\varphi}}[M(V)(x)]^{\frac{\alpha-1}{\alpha}}.$$

Proof. Let $\delta > 0$. Then

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\, dy = \int_{|x-y|<\delta} \frac{|V(y)|}{|x-y|^{n-1}}\, dy + \int_{|x-y|\geq\delta} \frac{|V(y)|}{|x-y|^{n-1}}\, dy. \tag{2.1}$$

Using Lemma (a) in [14], we have

$$\int_{|x-y|<\delta} \frac{|V(y)|}{|x-y|^{n-1}}\, dy \leq C(n)M(V)(x)\delta. \tag{2.2}$$

For the second term on the right hand side (2.1), let $q = n - \frac{p}{2}(\alpha+1)$, we use Hölder's inequality to obtain

$$\begin{aligned}\int_{|x-y|\ge\delta} \frac{|V(y)|}{|x-y|^{n-1}}\,dy &= \int_{|x-y|\ge\delta} \frac{|V(y)||x-y|^{\frac{q}{p}+1-n}}{|x-y|^{\frac{q}{p}}}\,dy \\ &\le \left(\int_{|x-y|\ge\delta} \frac{|V(y)|^p}{|x-y|^q}\,dy\right)^{\frac{1}{p}} \\ &\quad\times \left(\int_{|x-y|\ge\delta} |x-y|^{(\frac{q}{p}+1-n)(\frac{p}{p-1})}dy\right)^{\frac{p-1}{p}}. \qquad (2.3)\end{aligned}$$

Note that Nakai's condition gives us

$$\begin{aligned}\int_{|x-y|\ge\delta} \frac{|V(y)|^p}{|x-y|^q}\,dy &= \sum_{k=0}^{\infty} \int_{2^k\delta\le|x-y|<2^{k+1}\delta} \frac{|V(y)|^p}{|x-y|^q}\,dy \\ &\le C\|V\|^p_{L^{p,\varphi}} \int_\delta^\infty \frac{\varphi(t)}{t^{q+1}}\,dt \\ &\le C\|V\|^p_{L^{p,\varphi}}\delta^{n-p\alpha-q}. \qquad (2.4)\end{aligned}$$

Since $n + (\frac{q}{p}+1-n)(\frac{p}{p-1}) < 0$, we obtain

$$\int_{|x-y|\ge\delta} |x-y|^{(\frac{q}{p}+1-n)(\frac{p}{p-1})}dy = C(n,p,\alpha)\delta^{n+(\frac{q}{p}+1-n)(\frac{p}{p-1})}. \qquad (2.5)$$

Introducing (2.4) and (2.5) in (2.3), we have

$$\begin{aligned}\int_{|x-y|\ge\delta} \frac{|V(y)|}{|x-y|^{n-1}}\,dy &\le C\|V\|_{L^{p,\varphi}} \left(\delta^{n-p\alpha-q}\right)^{\frac{1}{p}} \left(\delta^{n+(\frac{q}{p}+1-n)(\frac{p}{p-1})}\right)^{\frac{p-1}{p}} \\ &= C\|V\|_{L^{p,\varphi}}\delta^{1-\alpha}. \qquad (2.6)\end{aligned}$$

From (2.6), (2.2) and (2.1), we get

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\,dy \le CM(V)(x)\delta + C\|V\|_{L^{p,\varphi}}\delta^{1-\alpha} \qquad (2.7)$$

For $\delta = \|V\|^{\frac{1}{\alpha}}_{L^{p,\varphi}}[M(V)(x)]^{-\frac{1}{\alpha}}$, the inequality (2.7) becomes

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\,dy \le C[M(V)(x)]^{1-\frac{1}{\alpha}}\|V\|^{\frac{1}{\alpha}}_{L^{p,\varphi}} = C[M(V)(x)]^{\frac{\alpha-1}{\alpha}}\|V\|^{\frac{1}{\alpha}}_{L^{p,\varphi}}.$$

Thus, the lemma is proved. □

Now, we are ready to prove Fefferman's inequality in generalized Morrey spaces.

Proof of Theorem 1.1. Let $1 < \gamma < p$ and $w := [M(|V|^\gamma)]^{\frac{1}{\gamma}}$. Then $w \in A_1 \cap L^{p,\varphi}$ according to Lemma 2.1. First, we will show that (1.5) holds for w in place of V. For any $u \in C_0^\infty(\mathbb{R}^n)$, let B be a ball such that $u \in C_0^\infty(B)$. From the well-known inequality

$$|u(x)| \le C\int_{B_0} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy, \qquad (2.8)$$

Tonelli's theorem, and Lemma 2.2, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |u(x)|^\alpha w(x)dx &= \int_B |u(x)|^\alpha w(x)dx \\ &\le C\int_B \left(\int_B \frac{|u(y)|^{\alpha-1}|\nabla u(y)|}{|x-y|^{n-1}}dy\right)|w(x)|dx \\ &\le C\|w\|_{L^{p,\varphi}}^{\frac{1}{\alpha}} \int_B |u(x)|^{\alpha-1}|\nabla u(x)|[M(w)(x)]^{\frac{\alpha-1}{\alpha}}dx. \qquad (2.9)\end{aligned}$$

Hölder's inequality and Lemma (2.1) imply that

$$\begin{aligned}\int_B |u(x)|^{\alpha-1}|\nabla u(x)|[M(w)(x)]^{\frac{\alpha-1}{\alpha}}\,dx &\le \left(\int_B |\nabla u(x)|^\alpha\,dx\right)^{\frac{1}{\alpha}} \\ &\quad\times\left(\int_B |u(x)|^\alpha M(w)(x)\,dx\right)^{\frac{\alpha-1}{\alpha}} \\ &\le C\left(\int_B |\nabla u(x)|^\alpha\,dx\right)^{\frac{1}{\alpha}} \\ &\quad\times\left(\int_B |u(x)|^\alpha w(x)\,dx\right)^{\frac{\alpha-1}{\alpha}}. \qquad (2.10)\end{aligned}$$

Substituting (2.10) into (2.9), we obtain

$$\int_{\mathbb{R}^n} |u(x)|^\alpha |w(x)|dx \le C\|w\|_{L^{p,\varphi}}^{\frac{1}{\alpha}}\left(\int_B |\nabla u(x)|^\alpha\,dx\right)^{\frac{1}{\alpha}}\left(\int_B |u(x)|^\alpha w(x)\,dx\right)^{\frac{\alpha-1}{\alpha}}.$$

Therefore

$$\int_{\mathbb{R}^n} |u(x)|^\alpha w(x)dx \le C\|w\|_{L^{p,\varphi}}\int_B |\nabla u(x)|^\alpha\,dx.$$

By (1.4), we have $|V(x)| = [|V(x)|^\gamma]^{\frac{1}{\gamma}} \le [M(|V(x)|^\gamma)]^{\frac{1}{\gamma}} = w(x)$. Hence, from the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces and Lemma 2.1, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} |u(x)|^\alpha |V(x)|dx &\le \int_{\mathbb{R}^n} |u(x)|^\alpha w(x)dx \\ &\le C\|w\|_{L^{p,\varphi}}\int_B |\nabla u(x)|^\alpha\,dx \\ &\le C\|V\|_{L^{p,\varphi}}\int_{\mathbb{R}^n} |\nabla u(x)|^\alpha\,dx.\end{aligned}$$

This completes the proof. □

We have already shown in Theorem 1.1 that Fefferman's inequality holds in generalized Morrey spaces under certain conditions.

2.2. **Fefferman's Inequality in Stummel Classes.** We need the following lemma to prove Fefferman's inequality where its potentials belong to Stummel classes. For the case $\alpha = 2$, this lemma can also be deduced from property of the Riez kernel which is stated in [17, p. 45].

Lemma 2.3. *Let $1 < \alpha \leq 2$ and $\alpha < n$. For any ball $B_0 \subset \mathbb{R}^n$, the following inequality holds:*

$$\int_{B_0} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\, dy \leq \frac{C}{|x-z|^{\frac{n-1}{\alpha-1}-1}}, \quad x, z \in B_0, \quad x \neq z.$$

Proof. Let $r := \frac{1}{2}|x-z|$. Then

$$\begin{aligned}\int_{B_0} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\, dy &\leq \sum_{j=2}^{\infty} \int_{2^j r \leq |x-y| < 2^{j+1} r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\, dy \\ &\quad + \int_{|x-y|<4r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\, dy \\ &= I_1 + I_2. \end{aligned} \tag{2.11}$$

For I_1, we get

$$\begin{aligned} I_1 &= \sum_{j=2}^{\infty} \int_{2^j r \leq |x-y| < 2^{j+1} r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\, dy \\ &\leq \sum_{j=2}^{\infty} \frac{1}{(2^j r)^{\frac{n-1}{\alpha-1}}} \int_{2^j r \leq |x-y| < 2^{j+1} r} \frac{1}{|z-y|^{n-1}}\, dy. \end{aligned} \tag{2.12}$$

Note that, $2^j r \leq |x-y| < 2^{j+1} r$ implies $2^j r \leq |x-y| < 2r + |z-y|$. Therefore $2^{j-1} r \leq 2^j r - 2r \leq |z-y|$. Hence the inequality (2.12) becomes

$$\begin{aligned} I_1 &\leq \sum_{j=2}^{\infty} \frac{1}{(2^j r)^{\frac{n-1}{\alpha-1}}} \int_{2^j r \leq |x-y| < 2^{j+1} r} \frac{1}{|z-y|^{n-1}}\, dy \\ &\leq C(n,\alpha) \sum_{j=2}^{\infty} \frac{1}{(2^j r)^{\frac{n-1}{\alpha-1}}} \frac{1}{(2^j r)^{n-1}} \int_{2^j r \leq |x-y| < 2^{j+1} r} 1\, dy \\ &\leq C(n,\alpha) \frac{1}{(r)^{\frac{n-1}{\alpha-1}-1}} \sum_{j=2}^{\infty} \frac{1}{(2^j)^{\frac{n-1}{\alpha-1}-1}}. \end{aligned} \tag{2.13}$$

Since $\frac{n-1}{\alpha-1} - 1 > 0$, the last series in (2.13) is convergent. This gives us

$$I_1 \leq C(n,\alpha) \frac{1}{(r)^{\frac{n-1}{\alpha-1}-1}} = \frac{C(n,\alpha)}{|x-z|^{\frac{n-1}{\alpha-1}-1}}. \tag{2.14}$$

For I_2, we obtain

$$I_2 = \int_{|x-y|<r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\, dy + \int_{r\le|x-y|<4r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\, dy$$
$$\le C(n,\alpha)\frac{1}{r^{\frac{n-1}{\alpha-1}-1}} = \frac{C(n,\alpha)}{|x-z|^{\frac{n-1}{\alpha-1}-1}}. \tag{2.15}$$

Combining (2.11), (2.14), and (2.15), the lemma is proved. □

The following theorem is Fefferman's inequality where the potential belongs to a Stummel class.

Proof of Theorem 1.2. The proof is separated into two cases, namely $\alpha = 1$ and $1 < \alpha \le 2$. We first consider the case $\alpha = 1$. Using the inequality (2.8) together with Fubini's theorem, we get

$$\int_{B_0} |u(x)||V(x)|^p\, dx \le C\int_{B_0} |\nabla u(y)| \int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}}\, dxdy$$
$$\le C\int_{B_0} |\nabla u(y)| \int_{B(y,2r_0)} \frac{|V(x)|^p}{|x-y|^{n-1}}\, dxdy.$$

It follows from the last inequality and the doubling property of Stummel p-modulus of V that

$$\int_{B_0} |u(x)||V(x)|^p\, dx \le C\,\eta_{\alpha,p}V(r_0)\int_{B_0} |\nabla u(x)|\, dx,$$

as desired.

We now consider the case $1 < \alpha \le 2$. Using the inequality (2.8) and Hölder's inequality, we have

$$\int_{B_0} |u(x)|^\alpha|V(x)|^p\, dx \le C\int_{B_0} |\nabla u(y)| \int_{B_0} \frac{|u(x)|^{\alpha-1}|V(x)|^p}{|x-y|^{n-1}}\, dx\, dy$$
$$\le C\left(\int_{B_0} |\nabla u(y)|^\alpha\right)^{\frac{1}{\alpha}}\left(\int_{B_0} F(y)^{\frac{\alpha}{\alpha-1}}\, dy\right)^{\frac{\alpha-1}{\alpha}}, \tag{2.16}$$

where $F(y) := \displaystyle\int_{B_0} \frac{|u(x)|^{\alpha-1}|V(x)|^p}{|x-y|^{n-1}}\, dx,\quad y\in B_0$. Applying Hölder's inequality again, we have

$$F(y) \le \left(\int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}}\, dx\right)^{\frac{1}{\alpha}}\left(\int_{B_0} \frac{|u(z)|^\alpha|V(z)|^p}{|z-y|^{n-1}}\, dz\right)^{\frac{\alpha-1}{\alpha}},$$

so that

$$\int_{B_0} F(y)^{\frac{\alpha}{\alpha-1}}\, dy \le \int_{B_0}\left(\int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}}\, dx\right)^{\frac{1}{\alpha-1}}\int_{B_0} \frac{|u(z)|^\alpha|V(z)|^p}{|z-y|^{n-1}}\, dz\, dy$$
$$= \int_{B_0} |u(z)|^\alpha|V(z)|^pG(z)\, dz, \tag{2.17}$$

where $G(z) := \int_{B_0} \left(\int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}|z-y|^{(n-1)(\alpha-1)}}\, dx \right)^{\frac{1}{\alpha-1}} dy$, $z \in B_0$. By virtue of Minkowski's integral inequality (or Fubini's theorem for $\alpha = 2$), we see that

$$G(z)^{\alpha-1} \le \int_{B_0} |V(x)|^p \left(\int_{B_0} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\, dy \right)^{\alpha-1} dx. \tag{2.18}$$

Combining (2.18), doubling property of Stummel p-modulus of V, and the inequality in Lemma 2.1, we obtain

$$G(z) \le C \left(\int_{B_0} \frac{|V(x)|^p}{|x-z|^{n-\alpha}}\, dx \right)^{\frac{1}{\alpha-1}} \le C[\eta_{\alpha,p} V(r_0)]^{\frac{p}{\alpha-1}}. \tag{2.19}$$

Now, (2.17) and (2.19) give

$$\int_{B_0} |F(y)|^{\frac{\alpha}{\alpha-1}}\, dy \le C[\eta_{\alpha,p} V(r_0)]^{\frac{p}{\alpha-1}} \int_{B_0} |u(x)|^\alpha |V(x)|^p\, dx. \tag{2.20}$$

Therefore, from (2.16) and (2.20), we get

$$\int_{B_0} |u(x)|^\alpha |V(x)|^p\, dx$$
$$\le C[\eta_{\alpha,p} V(r_0)]^{\frac{p}{\alpha}} \left(\int_{B_0} |\nabla u(x)|^\alpha\, dx \right)^{\frac{1}{\alpha}} \left(\int_{B_0} |u(x)|^\alpha |V(x)|^p\, dx \right)^{\frac{\alpha-1}{\alpha}}. \tag{2.21}$$

Dividing both sides by the third term of the right-hand side of (2.21), we get the desired inequality. □

Let B be an open ball in $\mathbb{R}^n$. If u has weak gradient ∇u in B and u is integrable over B, then we have the sub-representation inequality

$$|u(x) - u_B| \le C(n) \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy, \quad x \in B, \tag{2.22}$$

where $u_B := \frac{1}{|B|} \int_B u(y) dy$. Using the inequality (2.22) and the method in the proof of the previous theorem, we obtain the following result.

Theorem 2.4. *Let $1 \le p < \infty$, $1 \le \alpha \le 2$, and $\alpha < n$. Suppose that u has weak gradient ∇u in $B_0 := B(x_0, r_0) \subseteq \mathbb{R}^n$ and that u is integrable over B_0. If $V \in \tilde{S}_{\alpha,p}$, then*

$$\int_{B_0} |u(x) - u_{B(x_0,r_0)}|^\alpha |V(x)|^p dx \le C\, [\eta_{\alpha,p} V(r_0)]^p \int_{B_0} |\nabla u(x)|^\alpha dx,$$

where $C := C(n, \alpha)$.

Remark 2.5. Note that the case $\alpha = 2$ is exactly the Corollary 4.4 in [2].

3. Applications in Elliptic Partial Differential Equations

The two lemmas below tell us that if a function vanishes of infinity order at some $x_0 \in \Omega$ and fulfills the doubling integrability over some neighborhood of x_0, then the function must be identically to zero in the neighborhood.

Lemma 3.1 ([13]). *Let $w \in L^1_{\text{loc}}(\Omega)$ and $B(x_0, r) \subseteq \Omega$. Assume that there exists a constant $C > 0$ satisfying*

$$\int\limits_{B(x_0,r)} w(x)dx \leq C \int\limits_{B\left(x_0,\frac{r}{2}\right)} w(x)dx.$$

If w vanishes of infinity order at x_0, then $w \equiv 0$ in $B(x_0, r)$.

Lemma 3.2. *Let $w \in L^1_{\text{loc}}(\Omega)$ and $B(x_0, r) \subseteq \Omega$, and $0 < \beta < 1$. Assume that there exists a constant $C > 0$ satisfying*

$$\int\limits_{B(x_0,r)} w^\beta(x)dx \leq C \int\limits_{B\left(x_0,\frac{r}{2}\right)} w^\beta(x)dx.$$

If w vanishes of infinity order at x_0, then $w \equiv 0$ in $B(x_0, r)$.

Proof. According to the hypothesis, for every $j \in \mathbb{N}$ we have

$$\begin{aligned}
\int_{B(x_0,r)} w^\beta(x)dx &\leq C^1 \int_{B(x_0,2^{-1}r)} w^\beta(x)dx \\
&\leq C^2 \int_{B(x_0,2^{-2}r)} w^\beta(x)dx \\
&\quad \vdots \\
&\leq C^j \int_{B(x_0,2^{-j}r)} w^\beta(x)dx.
\end{aligned}$$

Hölder's inequality implies that

$$\left(\int_{B(x_0,r)} w^\beta(x)dx\right)^{\frac{1}{\beta}} \leq (C^{\frac{1}{\beta}})^j |B(x_0, 2^{-j}r)|^{\frac{1}{\beta}} \frac{|B(x_0, 2^{-j}r)|^k}{|B(x_0, 2^{-j}r)|^{k+1}} \int_{B(x_0,2^{-j}r)} w(x)dx. \tag{3.1}$$

Now we choose $k > 0$ such that $C^{\frac{1}{\beta}} 2^{-k} = 1$. Then (3.1) gives

$$\left(\int_{B(x_0,r)} w^\beta(x)dx\right)^{\frac{1}{\beta}} \leq (v_n r^n)^{\frac{1}{\beta}+k} (2^{-\frac{n}{\beta}})^j \frac{1}{|B(x_0, 2^{-j}r)|^{k+1}} \int_{B(x_0,2^{-j}r)} w(x)dx, \tag{3.2}$$

where v_n is the Lebesgue measure of unit ball in $\mathbb{R}^n$. Letting $j \to \infty$, we obtain from (3.2) that $w^\beta \equiv 0$ on $B(x_0, r)$. Therefore $w \equiv 0$ on $B(x_0, r)$. □

The following lemma is used by many authors in working with elliptic partial differential equation (for example, see [2, 3, 5, 30]). This lemma and the idea of its proof can be seen in [19]. We state and give the proof of this lemma, since it had never been stated and proved formally to the best of our knowledge.

Lemma 3.3. *Let $w : \Omega \to \mathbb{R}$ and $B(x, 2r)$ be an open ball in Ω. If $\log(w) \in BMO(B)$ with $B = B(x, r)$, then there exist $M > 0$ such that*

$$\int_{B(x,2r)} w^\beta dy \leq M^{\frac{1}{2}} \int_{B(x,r)} w^\beta dy$$

for some $0 < \beta < 1$, or

$$\int_{B(x,2r)} w \, dy \leq M^{\frac{1}{2}} \int_{B(x,r)} w \, dy.$$

Proof. By John-Nirenberg Theorem, there exist $\beta > 0$ and $M > 0$ such that

$$\left(\int_B \exp(\beta |\log(w) - \log(w)_B|) \, dy \right)^2 \leq M^2 |B|^2. \tag{3.3}$$

Assume that $\beta < 1$. Using (3.3), we compute

$$\begin{aligned}
&\left(\int_B w^\beta dy \right) \left(\int_B w^{-\beta} dy \right) \\
&= \left(\int_B \exp(\beta \log(w)) dy \right) \left(\int_B \exp(-\beta \log(w)) dy \right) \\
&= \left(\int_B \exp(\beta(\log(w) - \log(w)_B)) dy \right) \left(\int_B \exp(-\beta(\log(w) - \log(w)_B)) dy \right) \\
&\leq \left(\int_B \exp(\beta |\log(w) - \log(w)_B|) \, dy \right)^2 \leq M^2 |B|^2,
\end{aligned}$$

which gives

$$\left(\int_B w^{-\beta} dy \right)^{\frac{1}{2}} \leq M|B| \left(\int_B w^\beta dy \right)^{-\frac{1}{2}} \leq M|B| \left(\int_{B(x,2r)} w^\beta dy \right)^{-\frac{1}{2}}. \tag{3.4}$$

Applying Hölder's inequality and (3.4), we obtain

$$\begin{aligned}
|B| &\leq \int_B w^{\frac{\beta}{2}} u^{-\frac{\beta}{2}} dy \leq \left(\int_B w^\beta dy \right)^{\frac{1}{2}} \left(\int_B w^{-\beta} dy \right)^{\frac{1}{2}} \\
&\leq M|B| \left(\int_B w^\beta dy \right)^{\frac{1}{2}} \left(\int_{B(x,2r)} w^\beta dy \right)^{-\frac{1}{2}}.
\end{aligned} \tag{3.5}$$

From (3.5), we get

$$\int_{B(x,2r)} w^\beta dy \leq M^{\frac{1}{2}} \int_{B(x,r)} w^\beta dy.$$

For the case $\beta \geq 1$, we obtain from the inequality (3.3) that

$$\begin{aligned}
\left(\int_B \exp(|\log(w) - \log(w)_B|) \, dy \right)^2 &\leq \left(\int_B \exp(\beta |\log(w) - \log(w)_B|) \, dy \right)^2 \\
&\leq M^2 |B|^2.
\end{aligned}$$

Processing the last inequality with previously method, we get

$$\int_{B(x,2r)} w\,dy \le M^{\frac{1}{2}} \int_{B(x,r)} w\,dy.$$

The proof is completed. □

Theorems 1.1 and 1.2 are crucial in proving Theorem 1.4.

Proof of Theorem 1.4. Given $\delta > 0$ and let $\{u_k\}_{k=1}^{\infty}$ be a sequence in $C_0^{\infty}(\Omega)$ such that $\lim_{k\to\infty} \|u_k - u\|_{H^1(\Omega)} = 0$. By taking a subsequence, we assume $u_k + \delta \to u + \delta$ a.e. on Ω (see [1, p.94] or [8, p.29]) and $u_k + \delta > 0$ for all $k \in \mathbb{N}$, since $u \ge 0$.

Let $\psi \in C_0^{\infty}(B(x,2r))$, $0 \le \psi \le 1$, $|\nabla\psi| \le C_1 r^{-1}$, and $\psi := 1$ on $B(x,r)$. For every $k \in \mathbb{N}$, we have $\psi^{\alpha+1}/(u_k+\delta) \in H_0^1(\Omega)$. Using this as a test function in the weak solution definition (1.11), we obtain

$$\int_{\Omega} \langle a\nabla u, \nabla(u_k+\delta)\rangle \frac{\psi^{\alpha+1}}{(u_k+\delta)^2} = (\alpha+1)\int_{\Omega} \langle a\nabla u, \nabla\psi\rangle \frac{\psi^{\alpha}}{(u_k+\delta)} + \sum_{i=1}^{n}\int_{\Omega} b_i \frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u_k+\delta)} + \int_{\Omega} Vu\frac{\psi^{\alpha+1}}{(u_k+\delta)}. \tag{3.6}$$

Since $\operatorname{supp}(\psi) \subseteq B(x,2r)$, the inequality (3.6) reduces to

$$\int_{B(x,2r)} \langle a\nabla u, \nabla(u_k+\delta)\rangle \frac{\psi^{\alpha+1}}{(u_k+\delta)^2} = (\alpha+1)\int_{B(x,2r)} \langle a\nabla u, \nabla\psi\rangle \frac{\psi^{\alpha}}{(u_k+\delta)} + \sum_{i=1}^{n}\int_{B(x,2r)} b_i \frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u_k+\delta)} + \int_{B(x,2r)} Vu\frac{\psi^{\alpha+1}}{(u_k+\delta)}. \tag{3.7}$$

We will estimate all three terms on the right hand side of (3.7). For the first term, according to (1.7), we have

$$|\langle a\nabla u, \nabla\psi\rangle| \le \lambda^{-1}|\nabla u||\nabla\psi|. \tag{3.8}$$

Combining Young's inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$ for every $\epsilon > 0$ $(a, b > 0)$ and the inequality (3.8), we have for every $\epsilon > 0$

$$\begin{aligned}(\alpha+1)\int\limits_{B(x,2r)} \langle a\nabla u, \nabla\psi\rangle \frac{\psi^\alpha}{(u_k+\delta)} &\le \epsilon\lambda^{-1}(\alpha+1)\int\limits_{B(x,2r)} \frac{|\nabla u|^2}{(u_k+\delta)^2}\psi^{2\alpha} \\ &\quad + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon}\int\limits_{B(x,2r)} |\nabla\psi|^2 \\ &\le \epsilon\lambda^{-1}(\alpha+1)\int\limits_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2}\psi^{\alpha+1} \\ &\quad + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon}\int\limits_{B(x,2r)} |\nabla\psi|^2. \end{aligned} \tag{3.9}$$

To estimate the second term in (3.7), we use Hölder's inequality, Young's inequality and Theorem 1.1 or Theorem 1.2, to obtain

$$\begin{aligned}\int\limits_{B(x,2r)} b_i \frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u_k+\delta)} &\le \left(\int\limits_{B(x,2r)} \frac{|\nabla u|^2}{(u_k+\delta)^2}\psi^{\alpha+1}\right)^{\frac{1}{2}} \left(\int\limits_{B(x,2r)} b_i^2\psi^{\alpha+1}\right)^{\frac{1}{2}} \\ &\le \frac{\epsilon}{n}\int\limits_{B(x,2r)} \frac{|\nabla u|^2}{(u_k+\delta)^2}\psi^{\alpha+1} + \frac{1}{4n\epsilon}\int\limits_{B(x,2r)} b_i^2\psi^\alpha \\ &\le \frac{\epsilon}{n}\int\limits_{B(x,2r)} \frac{|\nabla u|^2}{(u_k+\delta)^2}\psi^{\alpha+1} + \frac{1}{4n\epsilon}C_1^i\int\limits_{B(x,2r)} |\nabla\psi|^\alpha. \end{aligned} \tag{3.10}$$

for every $i = 1, \dots, n$, where the constants C_1^i's depend on $n, \alpha, \|b_i^2\|_{L^{p,\varphi}}$ or $\eta_\alpha b_i^2(r_0)$. From (3.10) we have

$$\sum_{i=1}^n \int\limits_{B(x,2r)} b_i \frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u_k+\delta)} \le \epsilon\int\limits_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2}\psi^{\alpha+1} + \frac{1}{4\epsilon}C_2\int\limits_{B(x,2r)} |\nabla\psi|^\alpha, \tag{3.11}$$

where C_2 depends on $\max\limits_i\{C_1^i\}$. The estimate for the last term in (3.7) is

$$\int\limits_{B(x,2r)} Vu\frac{\psi^{\alpha+1}}{(u_k+\delta)} \le \int\limits_{B(x,2r)} V\frac{(u+\delta)}{(u_k+\delta)}\psi^\alpha. \tag{3.12}$$

Introducing (3.9), (3.11), and (3.12) in (3.7), we get

$$\begin{aligned}
&\int_{B(x,2r)} \langle a\nabla u, \nabla(u_k+\delta)\rangle \frac{\psi^{\alpha+1}}{(u_k+\delta)^2} \\
&\leq \epsilon\lambda^{-1}(\alpha+1)\int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2}\psi^{\alpha+1} + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon}\int_{B(x,2r)} |\nabla\psi|^2 \\
&\quad + \epsilon\int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2}\psi^{\alpha+1} + \frac{1}{4\epsilon}C_2\int_{B(x,2r)} |\nabla\psi|^\alpha + \int_{B(x,2r)} V\frac{(u+\delta)}{(u_k+\delta)}\psi^\alpha,
\end{aligned} \tag{3.13}$$

for every $k \in \mathbb{N}$.

Since $(u_k+\delta) \to (u+\delta)$ a.e. in Ω and $u+\delta > 0$, then

$$\frac{1}{(u_k+\delta)} \to \frac{1}{(u+\delta)}, \text{ a.e. in } \Omega. \tag{3.14}$$

For $j, i = 1, \ldots, n$, we infer from (3.14)

$$\frac{\partial(u+\delta)}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{1}{(u_k+\delta)^2} \to \frac{\partial(u+\delta)}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{1}{(u+\delta)^2}, \text{ a.e. in } B(x,2r). \tag{3.15}$$

For every $k \in \mathbb{N}$, we have

$$\left|\frac{\partial(u+\delta)}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{1}{(u_k+\delta)^2}\right| \leq \left|\frac{\partial(u+\delta)}{\partial x_j}\right|\left|\frac{\partial u}{\partial x_i}\right|\frac{1}{\delta^2}, \tag{3.16}$$

and

$$\int_{B(x,2r)} \left|\frac{\partial(u+\delta)}{\partial x_j}\right|\left|\frac{\partial u}{\partial x_i}\right|\frac{1}{\delta^2} \leq \frac{1}{\delta^2}\left\|\frac{\partial u}{\partial x_j}\right\|_{L^2(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\Omega)} < \infty, \tag{3.17}$$

since $u \in H_0^1(\Omega)$. The properties (3.15), (3.16), and (3.17) allow us to use the Lebesgue Dominated Convergent Theorem to obtain

$$\lim_{k\to\infty}\int_{B(x,2r)} \left|\frac{\partial(u+\delta)}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{1}{(u_k+\delta)^2} - \frac{\partial(u+\delta)}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{1}{(u+\delta)^2}\right| = 0. \tag{3.18}$$

By Hölder's inequality, we also have

$$\begin{aligned}
&\int_{B(x,2r)} \left|\left(\frac{\partial(u_k+\delta)}{\partial x_j} - \frac{\partial(u+\delta)}{\partial x_j}\right)\frac{\partial u}{\partial x_i}\frac{1}{(u_k+\delta)^2}\right| \\
&\leq \frac{1}{\delta^2}\left\|\frac{\partial(u_k+\delta)}{\partial x_j} - \frac{\partial(u+\delta)}{\partial x_j}\right\|_{L^2(B(x,2r))}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(B(x,2r))} \\
&\leq \frac{1}{\delta^2}\left\|\frac{\partial(u_k)}{\partial x_j} - \frac{\partial u}{\partial x_j}\right\|_{L^2(B(x,2r))}\|u\|_{H^1(\Omega)} \\
&\leq \frac{1}{\delta^2}\|u_k-u\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)}
\end{aligned} \tag{3.19}$$

for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} \|u_k - u\|_{H^1(\Omega)} = 0$, from (3.19) we get

$$\lim_{k\to\infty} \int\limits_{B(x,2r)} \left| \left(\frac{\partial(u_k+\delta)}{\partial x_j} - \frac{\partial(u+\delta)}{\partial x_j} \right) \frac{\partial u}{\partial x_i} \frac{1}{(u_k+\delta)^2} \right| = 0. \tag{3.20}$$

Note that

$$\begin{aligned}
&\int_{B(x,2r)} \left| a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial(u_k+\delta)}{\partial x_j} \frac{\psi^{\alpha+1}}{(u_k+\delta)^2} - a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial(u+\delta)}{\partial x_j} \frac{\psi^{\alpha+1}}{(u+\delta)^2} \right| \\
&\leq \frac{\|a_{ij}\|_{L^\infty(\Omega)}}{\delta^2} \int_{B(x,2r)} \left| \frac{\partial u}{\partial x_i} \frac{\partial(u_k+\delta)}{\partial x_j} \frac{1}{(u_k+\delta)^2} - \frac{\partial u}{\partial x_i} \frac{\partial(u+\delta)}{\partial x_j} \frac{1}{(u+\delta)^2} \right| \\
&\leq \frac{\|a_{ij}\|_{L^\infty(\Omega)}}{\delta^2} \int\limits_{B(x,2r)} \left| \frac{\partial(u+\delta)}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{1}{(u_k+\delta)^2} - \frac{\partial(u+\delta)}{\partial x_j} \frac{\partial u}{\partial x_i} \frac{1}{(u+\delta)^2} \right| \\
&\quad + \frac{\|a_{ij}\|_{L^\infty(\Omega)}}{\delta^2} \int\limits_{B(x,2r)} \left| \left(\frac{\partial(u_k+\delta)}{\partial x_j} - \frac{\partial(u+\delta)}{\partial x_j} \right) \frac{\partial u}{\partial x_i} \frac{1}{(u_k+\delta)^2} \right|,
\end{aligned} \tag{3.21}$$

for all $k \in \mathbb{N}$. Combining (3.18), (3.20), and (3.21), we have

$$\begin{aligned}
&\lim_{k\to\infty} \int_{B(x,2r)} \langle a\nabla u, \nabla(u_k+\delta)\rangle \frac{\psi^{\alpha+1}}{(u_k+\delta)^2} \\
&= \sum_{i,j=1}^{n} \lim_{k\to\infty} \int_{B(x,2r)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial(u_k+\delta)}{\partial x_j} \frac{\psi^{\alpha+1}}{(u_k+\delta)^2} \\
&= \sum_{i,j=1}^{n} \int_{B(x,2r)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial(u+\delta)}{\partial x_j} \frac{\psi^{\alpha+1}}{(u+\delta)^2} \\
&= \int_{B(x,2r)} \langle a\nabla u, \nabla(u+\delta)\rangle \frac{\psi^{\alpha+1}}{(u+\delta)^2}.
\end{aligned} \tag{3.22}$$

From (3.14),

$$\frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2} \psi^{\alpha+1} \to \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2} \psi^{\alpha+1}, \text{ a.e. in } B(x,2r). \tag{3.23}$$

For every $k \in \mathbb{N}$, we have

$$\frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2} \psi^{\alpha+1} \leq \frac{1}{\delta^2} |\nabla(u+\delta)|^2, \tag{3.24}$$

and

$$\int\limits_{B(x,2r)} \frac{1}{\delta^2} |\nabla(u+\delta)|^2 \leq \frac{1}{\delta^2} \|u\|_{H^1(\Omega)} < \infty, \tag{3.25}$$

since $u \in H_0^1(\Omega)$. Therefore, by (3.23), (3.24), (3.25), and Lebesgue Dominated Convergence Theorem,

$$\lim_{k\to\infty} \int\limits_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u_k+\delta)^2} \psi^{\alpha+1} = \int\limits_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2} \psi^{\alpha+1}. \tag{3.26}$$

We also have

$$V \frac{(u+\delta)}{(u_k+\delta)}\psi^\alpha \to V\frac{(u+\delta)}{(u+\delta)}\psi^\alpha = V\psi^\alpha, \text{a.e. in } B(x,2r) \tag{3.27}$$

because of (3.14). For every $k \in \mathbb{N}$, we have

$$\left| V \frac{(u+\delta)}{(u_k+\delta)}\psi^\alpha \right| \le \frac{1}{\delta}|V||u+\delta|. \tag{3.28}$$

If the assumption (1.8) holds, then

$$\begin{aligned} \int\limits_{B(x,2r)} \frac{1}{\delta}|V||u+\delta| &\le \frac{1}{\delta}\int\limits_{B(x,2r)} |V||u+\delta| \\ &< \frac{1}{\delta}\left(\int\limits_{B(x,2r)} |V|^2\right)^{\frac{1}{2}}\left(\int\limits_{B(x,2r)} |u+\delta|^2\right)^{\frac{1}{2}} < \infty, \end{aligned} \tag{3.29}$$

since $V \in L^2_{\text{loc}}(\mathbb{R})$ and $u \in H^1_0(\Omega)$. On the other hand, if the assumption (1.9) holds, then $V \in \tilde{S}_\alpha \subset \tilde{S}_1$ by virtue to [28, p.554]. Therefore, using Theorem 1.2 we have

$$\begin{aligned} \int\limits_{B(x,2r)} \frac{1}{\delta}|V||u+\delta| &\le \frac{1}{\delta}\int\limits_{B(x,2r)} |V| + \frac{1}{4\delta}\int\limits_{B(x,2r)} |V||u+\delta|^2 \\ &\le \frac{1}{\delta}\int\limits_{B(x,2r)} |V| + \frac{1}{4\delta}C(n)\eta_2 V(r)\int\limits_{B(x,2r)} |\nabla u|^2 < \infty, \end{aligned} \tag{3.30}$$

since $u \in H^1_0(\Omega)$. Combining (3.27), (3.28), (3.29) or (3.30), we can apply the Lebesgue Dominated Convergence Theorem to have

$$\lim_{k\to\infty} \int\limits_{B(x,2r)} V\frac{(u+\delta)}{(u_k+\delta)}\psi^\alpha = \int\limits_{B(x,2r)} V\psi^\alpha. \tag{3.31}$$

Theorem 1.1 or Theorem 1.2 allow us to get the estimate

$$\int\limits_{B(x,2r)} V\psi^\alpha \le C_3 \int\limits_{B(x,2r)} |\nabla\psi|^\alpha, \tag{3.32}$$

where the constant C_3 depends on n, α, and $\|V\|_{L^{p,\varphi}}$ or $\eta_\alpha V(r_0)$. Letting $k \to \infty$ in (3.13) and applying all informations in (3.22), (3.26), (3.31), and (3.32), we obtain

$$
\begin{aligned}
&\int_{B(x,2r)} \langle a\nabla u, \nabla(u+\delta)\rangle \frac{\psi^{\alpha+1}}{(u+\delta)^2} \\
&\le \epsilon\lambda^{-1}(\alpha+1) \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon} \int_{B(x,2r)} |\nabla\psi|^2 \\
&\quad + \epsilon \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} + \frac{1}{4\epsilon}C_2 \int_{B(x,2r)} |\nabla\psi|^\alpha + C_3 \int_{B(x,2r)} |\nabla\psi|^\alpha. \qquad (3.33)
\end{aligned}
$$

Notice that

$$
\lambda \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} \le \int_{B(x,2r)} \langle a\nabla u, \nabla(u+\delta)\rangle \frac{\psi^{\alpha+1}}{(u+\delta)^2}
$$

by the ellipticity condition (1.7). Moreover, by choosing $\epsilon := \frac{1}{2}\frac{\lambda^2}{(\alpha+1)+1}$, the inequality (3.33) is simplified by

$$
\int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} \le C_4 \int_{B(x,2r)} |\nabla\psi|^2 + C_5 \int_{B(x,2r)} |\nabla\psi|^\alpha, \qquad (3.34)
$$

where the constant C_4 depends on α and λ, while the constant C_5 depends on C_2 and C_3. Therefore, (3.34) implies

$$
\begin{aligned}
\int_{B(x,r)} |\nabla \log(u+\delta)|^2 &\le \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} \\
&\le C_5 \int_{B(x,2r)} |\nabla\psi|^2 + C_6 \int_{B(x,2r)} |\nabla\psi|^\alpha \\
&\le C\left(r^{-2}r^n + r^{-\alpha}r^n\right) = Cr^{-2}r^n.
\end{aligned}
$$

The last constant C depends on C_4 and C_5. From Hölder's inequality,

$$
\left(\frac{1}{r^n}\int_{B(x,r)} |\nabla \log(u+\delta)|^\alpha\right)^{\frac{2}{\alpha}} \le \frac{1}{r^n}\int_{B(x,r)} |\nabla \log(u+\delta)|^2 \le Cr^{-2},
$$

whence

$$
\frac{1}{r^n}\int_{B(x,r)} |\nabla \log(u+\delta)|^\alpha \le Cr^{-\alpha}. \qquad (3.35)
$$

By using Poincaré's inequality together with the inequality (3.35), the theorem is proved. □

By virtue of Theorem 1.4, we have the following corollary.

Corollary 3.4. *Let $u \ge 0$ be a weak solution of $Lu = 0$ and $B(x,2r) \subseteq \Omega$ where $r \le 1$. Then, for every $\delta > 0$, $\log(u+\delta) \in BMO_\alpha(B(x,r))$.*

Gathering Lemma 3.1, Lemma 3.2, Lemma 3.3, and Corollary 3.4, we obtain the unique continuation property of the equation $Lu = 0$ stated in Corollary 1.5.

Proof of the Corollary 1.5. Given $x \in \Omega$ and let $B := B(x,r)$ be a ball where $B(x,2r) \subseteq \Omega$ and $r \leq 1$. Let $\{\delta_j\}$ be a sequence of real numbers in $(0,1)$ which converges to 0. From Corollary 3.4, we get $\log(u+\delta_j) \in BMO_\alpha(B)$. Therefore $\log(u+\delta_j) \in BMO(B)$. According to Lemma 3.3, there exists a constant $M > 0$ such that we have two cases:

$$\int_{B(x,2r)} u^\beta dy \leq \int_{B(x,2r)} (u+\delta_j)^\beta dy \leq M^{\frac{1}{2}} \int_{B(x,r)} (u+\delta_j)^\beta dy,$$

where $0 < \beta < 1$, or,

$$\int_{B(x,2r)} u\, dy \leq \int_{B(x,2r)} (u+\delta_j)\, dy \leq M^{\frac{1}{2}} \int_{B(x,r)} (u+\delta_j)\, dy.$$

In both cases, letting $j \to \infty$, we obtain

$$\int_{B(x,2r)} u^\beta dy \leq M^{\frac{1}{2}} \int_{B(x,r)} u^\beta dy,$$

or,

$$\int_{B(x,2r)} u\, dy \leq M^{\frac{1}{2}} \int_{B(x,r)} u\, dy.$$

Therefore, using Lemma 3.2 for the first case and Lemma 3.1 for the second case, $u \equiv 0$ in $B(x,2r)$ if u vanishes of infinity order at x. □

The example below shows that there exist an elliptic partial differential which does not satisfies strong unique continuation property where its potential belongs to Morrey spaces $L^{p,n-4p}$ and $\tilde{S}_\beta$ for all $\beta \geq 4$.

Example 3.5. Let $\Omega = B(0,1) \subseteq \mathbb{R}^n$, $w : \Omega \to \mathbb{R}$ and $V : \mathbb{R}^n \to \mathbb{R}$ which are defined by the formula

$$w(x) = \begin{cases} \exp(-|x|^{-1})|x|^{-(n+1)} & , x \in \Omega\backslash\{0\} \\ 1 & , x = 0, \end{cases}$$

and

$$V(x) = \begin{cases} 3(n+1)|x|^{-2} - (n+5)|x|^{-3} + |x|^{-4} & , x \in \mathbb{R}^n\backslash\{0\} \\ 0 & , x = 0. \end{cases}$$

Note that, $w(x) > 0$ for all $x \in \Omega$. We will show that w vanishes of infinity order at $x = 0$ and is a solution of Schrödinger equation

$$-\Delta w(x) + V(x)w(x) = 0, \qquad x \in \Omega. \tag{3.36}$$

We calculate

$$\int_{|x|<r} w(x)dx = \int_{|x|<r} \exp(-|x|^{-1})|x|^{-(n+1)}dx$$
$$= C(n)\exp\left(-\frac{1}{r}\right),$$

by using polar coordinate. From the fact

$$\lim_{r\to 0} \exp\left(-\frac{1}{r}\right)(r)^{-\gamma} = 0,$$

for all $\gamma > 0$, then

$$\lim_{r\to 0} \frac{1}{|B(0,r)|^\kappa} \int_{|x|<r} w(x)dx = C(n) \lim_{r\to 0} \exp\left(-\frac{1}{r}\right) \frac{1}{r^{n\kappa}} = 0,$$

for all $\kappa > 0$. Therefore w vanishes of infinity order at $x = 0$. For $i = 1, \ldots, n$,

$$\frac{\partial w}{\partial x_i}(x) = w(x) \left(-(n+1)|x|^{-2} x_i + |x|^{-3} x_i\right).$$

Hence

$$\frac{\partial^2 w}{\partial x_i^2}(x) = w(x) g(x),$$

where

$$\begin{aligned} g(x) = &-(n+1)|x|^{-2} + |x|^{-3} + (2(n+1) + (n+1)^2)|x|^{-4} x_i^2 \\ &- (3 + 2(n+1))|x|^{-5} x_i^2 + |x|^{-6} x_i^2. \end{aligned}$$

Consequently

$$\begin{aligned} \Delta w(x) &= \sum_{i=1}^n \frac{\partial^2 w}{\partial x_i^2}(x) = \sum_{i=1}^n w(x) g(x) \\ &= w(x) \sum_{i=1}^n g(x) = w(x) \left(3(n+1)|x|^{-2} - (n+5)|x|^{-3} + |x|^{-4}\right) \\ &= w(x) V(x). \end{aligned}$$

This shows that w is the solution of the Schrödinger equation. We conclude that this Schrödinger equation does not satisfy the strong unique continuation property since the solution w vanishes of infinity order at $x = 0$, but strictly positive function in Ω. Now, we will analyze the property of function V.

Let $y \in \mathbb{R}^n$ and $y \neq 0$, we get

$$|V(y)| \leq 3(n+1)|y|^{-2} + (n+5)|y|^{-3} + |y|^{-4}. \tag{3.37}$$

For every $\beta > 4$ and $x \in \mathbb{R}^n$, then

$$\begin{aligned} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-\beta}} dy \leq C(n) \int_{|x-y|<r} \frac{|y|^{-2}}{|x-y|^{n-\beta}} dy + C(n) \int_{|x-y|<r} \frac{|y|^{-3}}{|x-y|^{n-\beta}} dy \\ + C(n) \int_{|x-y|<r} \frac{|y|^{-4}}{|x-y|^{n-\beta}} dy, \end{aligned} \tag{3.38}$$

by using (3.37). For $m \in \{2, 3, 4\}$, we estimate

$$\begin{aligned} \int_{|x-y|<r} \frac{|y|^{-m}}{|x-y|^{n-\beta}} dy &= \int_{\substack{\{|x-y|<|y|\} \\ \cap \{|x-y|<r\}}} \frac{|y|^{-m}}{|x-y|^{n-\beta}} dy + \int_{|y|\leq|x-y|<r} \frac{|y|^{-m}}{|x-y|^{n-\beta}} dy \\ &\leq \int_{|x-y|<r} \frac{|x-y|^{-m}}{|x-y|^{n-\beta}} dy + \int_{|y|<r} \frac{|y|^{-m}}{|y|^{n-\beta}} dy \\ &= C(n, m, \beta) r^{\beta-m}. \end{aligned} \tag{3.39}$$

Introducing (3.39) in (3.38) we obtain

$$\int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-\beta}} dy \leq C(n,\beta)(r^{\beta-2} + r^{\beta-3} + r^{\beta-4}) \tag{3.40}$$

Since x arbitrary in (3.40), then

$$\eta_\beta V(r) \leq C(n,\beta)(r^{\beta-2} + r^{\beta-3} + r^{\beta-4})$$

which tell us $\eta_\beta V(r) \to 0$ for $r \to 0$. Whence $V \in S_\beta \subseteq \tilde{S}_\beta$ for all $\beta > 4$. Moreover, $\eta_\beta V(r) < \infty$ for $\beta = 4$ and hence $V \in \tilde{S}_4$.

Let $x \neq 0$ and $|x| \leq (n+5)^{-1}$, we have $-(n+5)|x|^{-3} + |x|^{-4} \geq 0$. This implies

$$V(x) \geq 3(n+1)|x|^{-2}. \tag{3.41}$$

Given $1 \leq \alpha \leq 2$ and $0 < r < (n+5)^{-1}$, then by (3.41)

$$\begin{aligned} \eta_\alpha V(r) &\geq C(n) \int_{|y|<r} \frac{|y|^{-2}}{|y|^{n-\alpha}} dy \\ &\geq C(n) r^{\alpha-2} \int_{|y|<r} \frac{1}{|y|^n} dy = \infty. \end{aligned} \tag{3.42}$$

Thus $V \notin \tilde{S}_\alpha$.

Define a function $V^* = V\chi_\Omega$. Then $V^* : \mathbb{R}^n \to \mathbb{R}$ and w is a solution the equation (3.36) where V is replaced by V^*. For $y \in \mathbb{R}^n$ and $y \neq 0$, we get

$$|V^*(y)| \leq (4n+9)|y|^{-4}. \tag{3.43}$$

Given $x \in \mathbb{R}^n$ and $r > 0$. By (3.43) and using similar technique as in (3.39), we have

$$\frac{1}{r^{n-4p}} \int_{|x-y|<r} |V^*(y)|^p dy \leq \frac{1}{r^{n-4p}} \int_{|x-y|<r} |y|^{-4p} dy = C(n,p). \tag{3.44}$$

According to (3.44), we conclude $V^* \in L^{p,n-4p}$. □

Remark 3.6. The equation $Lu = 0$ has the strong unique continuation property if $V, b_i^2 \in \tilde{S}_\alpha$ for $i = 1, \ldots, n$ and $1 \leq \alpha \leq 2$ (see assumption (1.9)). In view of Example 3.5, there exist $V \in \tilde{S}_\alpha$, $\alpha \geq 4$, and $b_i = 0$ for $i = 1, \ldots, n$ such that the equation $Lu = 0$ does not have the strong unique continuation property. However, the authors still do not know whether $Lu = 0$ has the strong unique continuation property or not if $V, b_i^2 \in \tilde{S}_\alpha$ for $i = 1, \ldots, n$ and $2 < \alpha < 4$.

Remark 3.7. The equation $Lu = 0$ has the strong unique continuation property if $V, b_i^2 \in L^{p,\varphi}$ where (1.8) holds. If we choose $V \in L^{p,n-4p}$ (i.e. $\alpha = 4$) as in Example 3.5 and $b_i = 0$ for $i = 1, \ldots, n$, then the equation $Lu = 0$ does not have the strong unique continuation property.

Acknowledgement. This research is supported by ITB Research & Innovation Program 2019. The first author also thanks LPDP Indonesia.

References

1. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York Dordrecht Heidelberg London, (2010)
2. Castillo, R.E., Ramos-Fernández, J., Rojas, E.M.: A note on the Kato class and some applications. Positivity. **23(2)**, 327–356 (2019)
3. Chiarenza, F., Fabes, E., Garofalo, N.: Harnack's inequality for Schrödinger operators and the continuity of solutions. Proc. Amer. Math. Soc. **98(3)**, 415–425 (1986)
4. Chiarenza, F., Frasca, M.: A remark on a paper by C. Fefferman. Proc. Amer. Math. Soc. **108**, 407–409 (1990)
5. Chiarenza, F., Garofalo, N.: Unique continuation for nonnegative solutions of Schrödinger operators. Institute for Mathematics and its Applications, Preprint Series No 122, University of Minnesota (1984)
6. Davis, E.B., Hinz, A.: Kato class potentials for higher order elliptic operators. J. London Math. Soc. **58**, 669–678 (1998)
7. Evans, L.C.: Partial Differential Equations. 2nd ed. Graduate Studies in Mathematics, **19**, American Mathematical Society Providence, Rhode Island (2010)
8. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Property of Functions. Revised Edition. CRC Press, Boca Raton (2015)
9. Fabes, E., Garofalo, N., Lin, F.: A partial answer to a conjecture of B. Simon concerning unique continuation. J. Funct. Anal. **88(1)**, 194–210 (1990)
10. Fefferman, C.: The uncertainty principle. Bull. Amer. Math. Soc. **9**, 129–206 (1983)
11. Garcia-Cuerva, J., Rubio de Francia, J.L.: Weighted Norm Inequalities and Related Topic. North Holland, Amsterdam New York Oxford, (1985)
12. Garofalo, N., Lin, F.H.: Unique continuation for elliptic operators: a geometric-variational approach. Commun. Pure Appl. Math. **40(3)**, 347–366 (1987)
13. Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Princeton University Press, New Jersey (1983)
14. Hedberg, L.I.: On certain convolution inequalities. Proc. Amer. Math. Soc. **36**, 505–510 (1972)
15. Jerison, D., Kenig, C.E.: Unique continuation and absence of positive eigenvalues for Schrödinger operators. Ann. Math. **121(3)**, 463-488, (1985)
16. Koch, H., Tataru, D.: Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients. Commun. Pure Appl. Math. **54(3)** , 339-360 (2001)
17. Landkof, N. S.: Fondations of Modern Potential Theory. Springer-Verlag, Berlin Heidelberg New York, (1972)
18. Morrey, C.B.: On the solutions of quasi-linear elliptic partial differential equations. Trans. Amer. Math. Soc. **43**, 126–166 (1938)
19. Moser, J.: On Harnack's theorem for elliptic differential equations.. Comm. Pure Appl. Math. **14**, 577–591 (1961)
20. Nakai, E.: Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces. Math. Nachr. **166**, 95–103 (1994)
21. Nakai, E.: Generalized fractional integrals on generalized Morrey spaces. Math. Nachr. **287(2)**, 339–351 (2014)
22. Pick, L., Kufner, A., John, O., Fucik, S.: Function Spaces. Vol. 1. 2nd Revised and Extended Version. De Gruyter, Berlin (2013)
23. Ragusa, M.A., Zamboni, P.: A Potential Theoretic Inequality. Czech. Mat. J. **51**, 55–56 (2001)
24. Sawano, Y.: Generalized Morrey spaces for non-doubling measures. Nonlinear Differ. Equ. Appl. **15**, 413–425 (2008)
25. Seo, I.: On minimal support properties of solutions of Schrödinger equations. J. Math. Anal. Appl. **414(1)**, 21–28 (2014)

26. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton New Jersey (1970)
27. Tumalun, N. K., Gunawan, H.: Morrey spaces are embedded between weak Morrey spaces and Stummel classes. J. Indones. Math. Soc. **25 (3)**, 203–209 (2019)
28. Tumalun, N.K., Hakim, D.I., Gunawan, H.: Inclusion between generalized Stummel classes and other function spaces. Math. Inequal. Appl. **23(2)**, 547–562 (2020)
29. Wheeden, R.L., Zygmund, A.: Measure and Integral, An Introduction to Real Analysis, Second Edition. CRC Press, Boca Raton London New York (2015)
30. Zamboni, P.: Some function spaces and elliptic partial differential equations. Le Matematiche **42(1)**, 171–178 (1987)

[1]Analysis and Geometry Group, Bandung Institute of Technology, Jl. Ganesha No. 10, Bandung 40132, Indonesia
E-mail address: nickytumalun@yahoo.co.id

[2]Analysis and Geometry Group, Bandung Institute of Technology, Jl. Ganesha No. 10, Bandung 40132, Indonesia
E-mail address: dhakim@math.itb.ac.id

[3]Analysis and Geometry Group, Bandung Institute of Technology, Jl. Ganesha No. 10, Bandung 40132, Indonesia
E-mail address: hgunawan@math.itb.ac.id